Ultra-low threshold current density quantum dot lasers using the dots-in-a-well (DWELL) structure

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ABSTRACT

Quantum dots laser diodes using the dots-in-a-well (DWELL) structure (InAs dots in an InGaAs quantum wells) have exhibited significant recent progress. With a single InAs dot layer in $In_{0.15}Ga_{0.85}As$ quantum well, threshold current densities are as low as 26 A cm^{-2} at 1.25 μm. Quantum dot laser threshold current densities are now lower than any other reported semiconductor laser. In this work, the threshold current density is reduced to 16 A cm^{-2} by HR coatings on the same device. Further investigation of performance reveals that use of multiple DWELL stacks improves the modal gain and internal quantum efficiency. It is suggested that carrier heating out of the quantum dots limits the T_0 value of these DWELL lasers.

Keywords: Quantum dot lasers, low threshold current density, modal gain.

1. INTRODUCTION

While it has been long predicted that the reduction in density of states should lead to lower threshold current densities for quantum dot lasers over quantum well lasers [1], this has only recently been demonstrated by our group [2]. This breakthrough have come after many years of efforts to reduce the threshold current density of quantum dot lasers on GaAs substrates [3, 4, 5, 6, 7, 8].

Our approach to threshold reduction has been to place the InAs quantum dots in a strained InGaAs quantum well (dots-in-a-well, or "DWELL") [2, 3]. As a result, an extremely low threshold current density, 26 A cm^{-2}, is achieved with a single layer of InAs quantum dots in a strained $In_{0.15}Ga_{0.85}As$ quantum well [2]. This result was the first time that the threshold current density performance of quantum dot lasers surpassed quantum well lasers [9, 10]. However in that work, only 7.8 mm long, cleaved-facet lasers were studied which resulted in a relatively high mirror loss (~1.46cm^{-1}). Since the cavity internal loss may be very low [3], the ultimate threshold current density could only be found if the mirror loss were reduced. In this work, the lasers reported in Ref. [2] are characterized with HR coatings on both facets to reduce mirror loss.

Additionally, one problem with the quantum dot lasers reported in Ref [2] is the low ground state modal gain of about 3.5cm^{-1}. Excited state emission occurs when the cavity length is reduced to increase the cavity loss (Fig.1.). For cavity lengths longer than 4 mm, pure ground state lasing can be observed at threshold. When the current is further increased to five times of threshold, significant contribution from the first excited state can be observed. In this work, a three DWELL-stack laser is designed and grown with the goal of improving the ground state modal gain.

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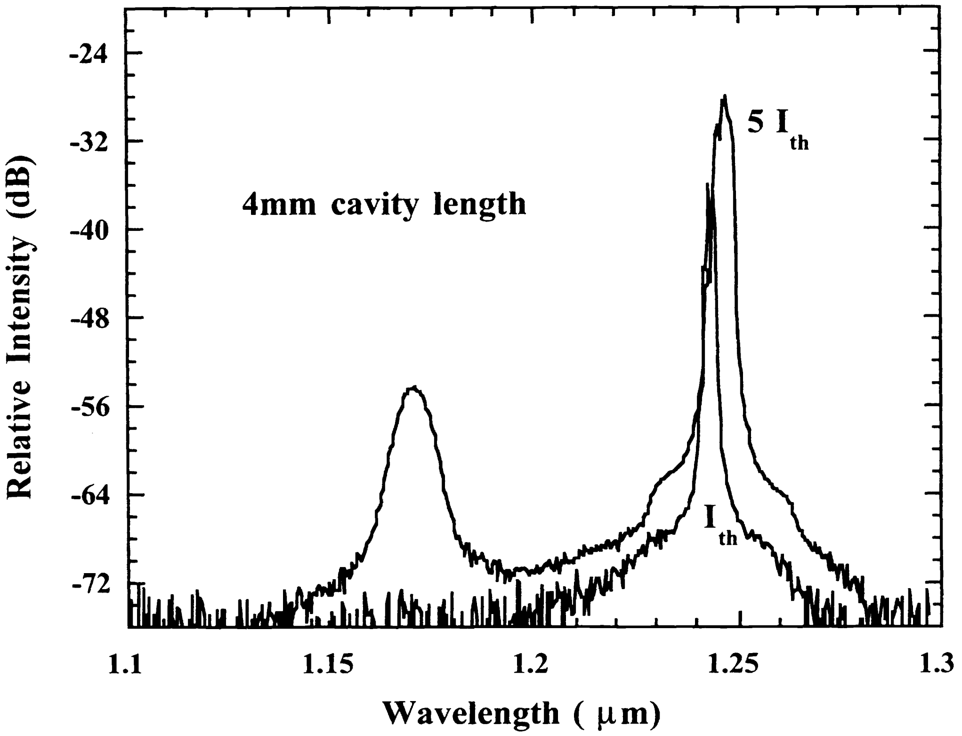

Fig.1. At a cavity length of 4mm, pure ground state lasing can be observed at threshold. When the current is further increased to five times of threshold, significant contribution from the first excited state can be observed.

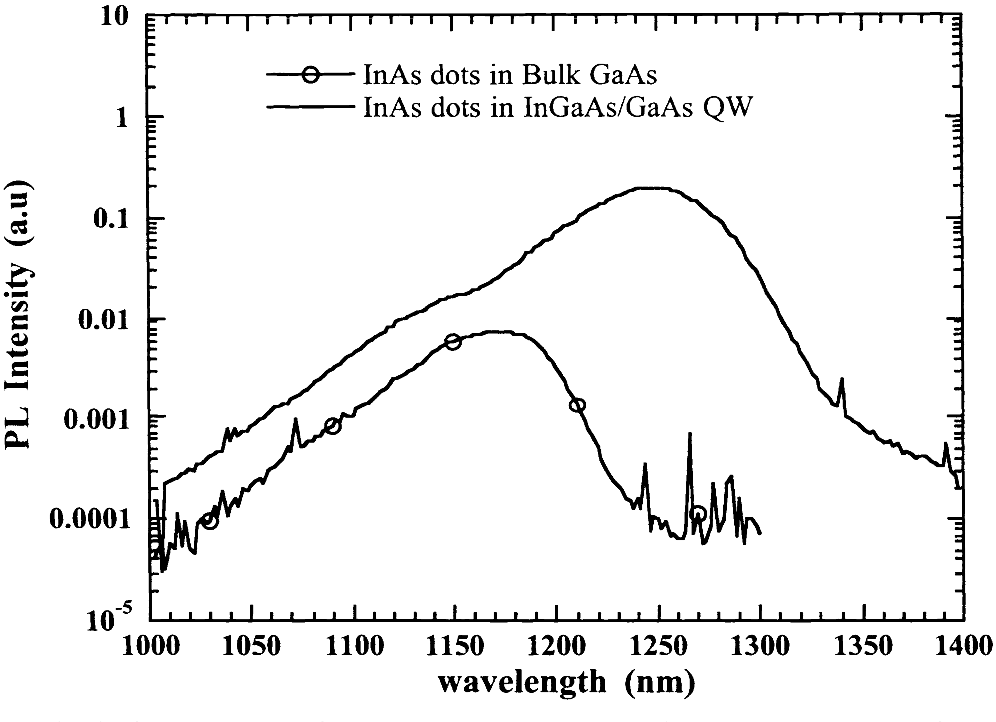

Fig.2. Photoluminesce experiments for InAs quantum dots grown in two difference material matrixes, GaAs bulk material and InGaAs/GaAs strained quantum well under identical growth conditions. The PL intensity is an order of magnitude stronger when dots were grown in InGaAs quantum well. The wavelength is also longer.

In the previous work [2], a low T_0 value characterizing the temperature dependence of the threshold current density was reported. It was originally thought that the quantum well is limiting the T_0 value of the DWELL structure. In this work, a quantum well laser that is identical to the 1-DWELL laser is grown and is used as a reference to understand the factors limiting the T_0 value of the DWELL lasers.

2. PHYSICS OF DWELL STRUCTURE

The major benefit of this "dots-in-a-well" (DWELL) design can be illustrated by photoluminescence results. A single layer of quantum dots was grown under otherwise identical growth conditions in two different materials, bulk GaAs and an InGaAs/GaAs strained quantum well. Photoluminescence was studied using a He-Ne laser and the results are shown in Fig. 2. It can be seen from the plot that the photoluminescence intensity is much stronger when quantum dots were grown in the InGaAs quantum well. The emission wavelength is also longer, and with simple modifications to the growth process, it is possible to fabricate dots that emit at wavelengths longer than 1.3 μm. It is also observed that the dot density is higher (densities up to $7\text{x}10^{10}$ cm^{-2}) when dots were grown in InGaAs quantum well [3] than when grown in bulk GaAs (densities around $2\text{x}10^{10}$ cm^{-2})

An important advantage of the DWELL structure can be understood by comparing the fill factor of InAs quantum dots grown in bulk GaAs and in InGaAs quantum well (Fig.3.). If we assume a dot density of 10^{10}cm^{-2} and dot size of about 20 nm, the areal fill factor is only about 4%. In the simplest approximation, This means that 96% of the injected carriers bypass the dots. When dots are embedded into a quantum well, the quantum well helps to capture and confine carrier motion to a 2-D plane. Even though they may miss one dot, they will eventually be captured by another dot.

Previous concerns with the DWELL structure included the possibility that the quantum well radiative transition may compete with the quantum dot radiative transition [11]. No quantum well emission was observed in this work. This implies that the carrier capture time of the quantum dots is much shorter than the quantum well radiative lifetime.

3. DEVICE STRUCTURES AND GROWTH

The laser design for the 1-DWELL laser is shown in Fig.4. The laser structures were all grown by solid-source molecular beam epitaxy (MBE) on a n^+ GaAs substrate. The epitaxial structure consists of an n-type (10^{18} cm^{-3}) 300 nm thick GaAs buffer, a 2 μm n-type (10^{17}cm^{-3}) lower $Al_{0.7}Ga_{0.3}As$ cladding layer, a 230 nm thick GaAs waveguide surrounding the laser active region, a 2 μm p-type (10^{17}cm^{-3}) upper cladding layer, and a p^+–doped ($3\text{x}10^{19}$ cm^{-3}) 60 nm thick GaAs cap.

In the center of the waveguide, an equivalent coverage of 2.4 monolayers of InAs results in quantum dots grown approximately in the middle of the 100 Å $In_{.15}Ga_{.85}As$ quantum well. The quantum dots and quantum well were grown at 510 °C, and all other layers were grown at 610 °C, as measured by an optical pyrometer. The photoluminescence linewidth is 37 meV. The dot density is $3.2\text{x}10^{10}$cm^{-2}.

The $In_{0.15}Ga_{0.85}As$ 3-DWELL stack laser is the same as 1-DWELL stack laser except that there are three quantum wells separated by 10nm thick barriers with one dot layer in each well. The $In_{0.15}Ga_{0.85}As$ single QW laser is the same as one $In_{0.15}Ga_{0.85}As$ DWELL stack laser except without the quantum dot layer.

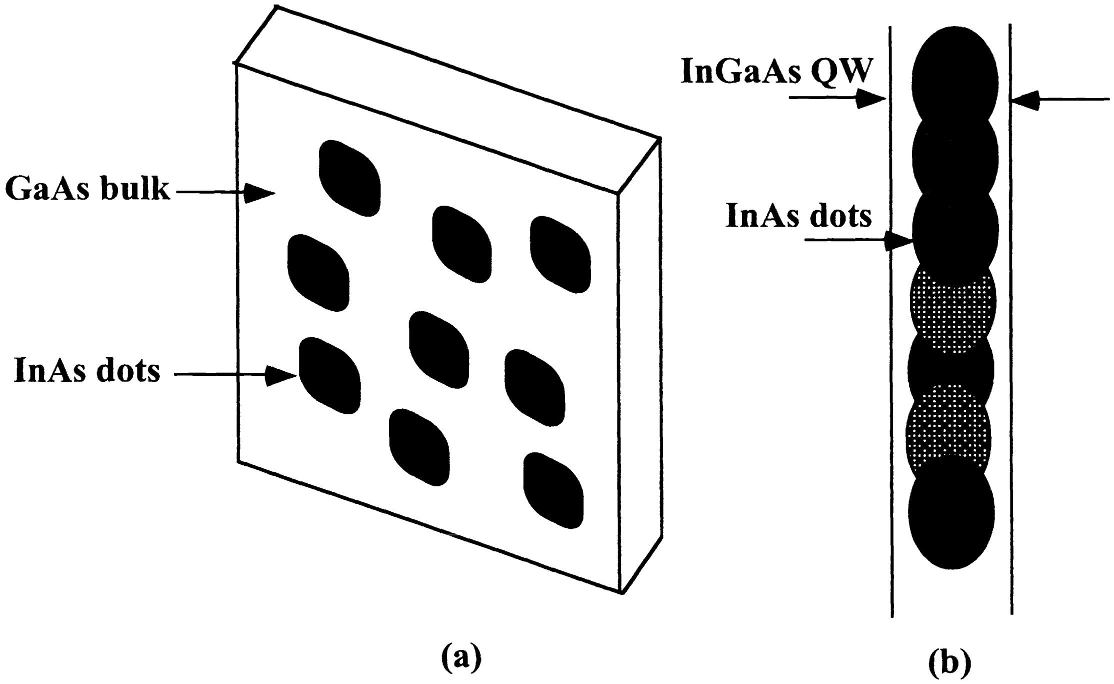

Fig.3. Fill factor (FF) comparison of InAs quantum dots grown (a) bulk GaAs (FF = 4%, assuming $10^{10}cm^{-2}$ dot density and ~20nm dot size). and (b) InGaAs quantum well (FF is much improved).

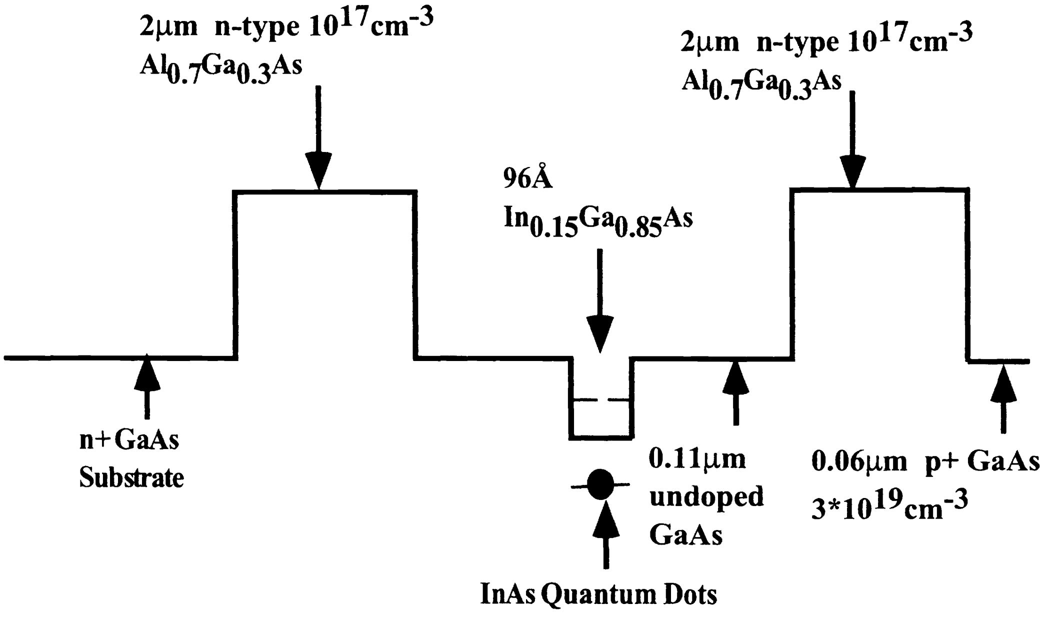

Fig.4. 1-DWELL laser design

3. RESULTS

Broad area lasers with 100 μm stripe widths were fabricated from these three laser structures. The wafers were then cleaved into laser bars of different cavity lengths. The longest cleaved facet laser bars are 7.8mm. All devices were tested with the epi-side up on a thermoelectric cooler using pulsed excitation. The pulse width was 300 ns with a duty cycle of 0.5%. The temperature of the thermoelectric cooler was set to be 20 °C.

3.1 Threshold Current density

The L-I curve of the 1-DWELL and 1-QW 7.8mm cleaved facet lasers are plotted in Fig.5. The low threshold current of the DWELL laser is clearly due to the low transparency current of quantum dot material. The lowest threshold current density achieved by the 7.8mm cleaved facet laser bars are 26, 36, and 188 A cm^{-2} for the 1-DWELL laser, 3-DWELL laser and 1-QW laser, respectively. The mirror losses of the 1-DWELL laser and 3-DWELL are further reduced by HR coating on two facets. The lowest threshold current density achieved by the 1-DWELL laser is 16 A cm^{-2} with 75% and 98% HR coatings on both sides. The lowest threshold current density achieved by the 3-DWELL laser is 31 A cm^{-2} with 98% HR coatings on one side. The near field pattern of the $In_{0.15}Ga_{0.85}As$ 1-DWELL laser is also measured at threshold. Very uniform lasing from the whole stripe cross section is observed. The fact that this kind of laser has very small linewidth enhancement factor [12] implies that the filamentation effect is very small for these lasers and extremely low threshold current density we have measured is a real value.

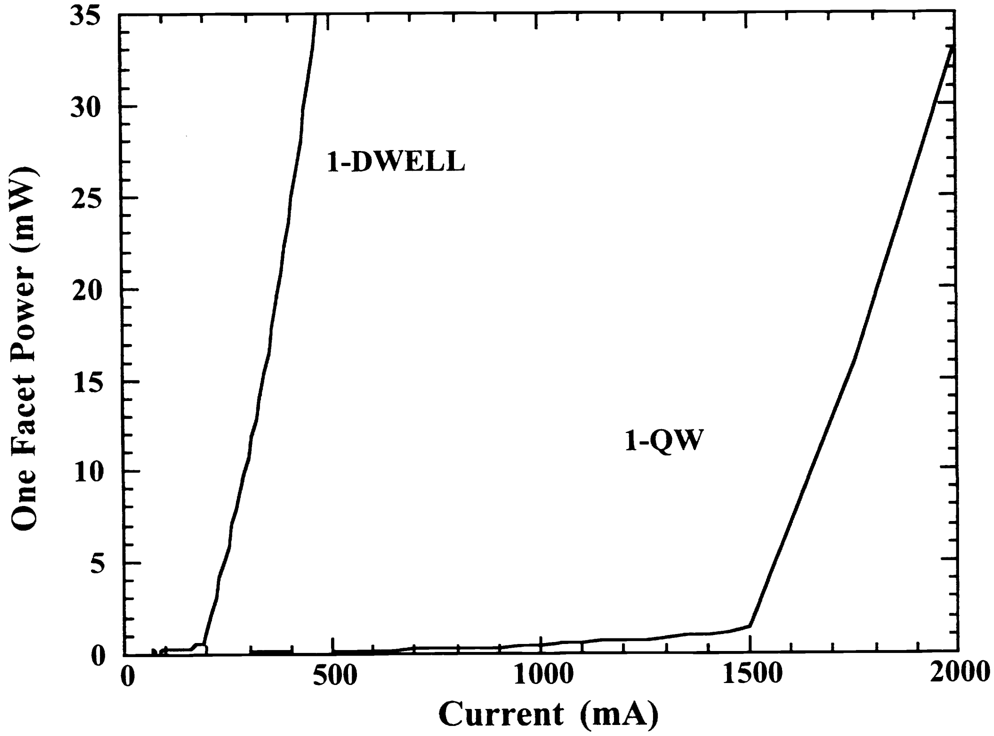

Fig.5. The L-I curve of the 1-DWELL and 1-QW 7.8mm cleaved facet lasers

3.2. Internal quantum efficiency and internal loss

The inverse external quantum efficiency, $1/\eta_{ex}$, versus cavity length, L, results for the 1-DWELL and 3-DWELL lasers in Fig.6. are used to calculate the internal loss, α_i, and the internal quantum efficiency, η_i. By fitting the data to the equation $1/\eta_{ex} = 1/\eta_i(1- \alpha_i L/\ln(R))$, η_i (α_i) values of 38.1% (0.63 cm^{-1}) and 52.7% (1.17 cm^{-1}) are found for the 1-DWELL and 3-DWELL lasers, respectively. The reflectivity, *R*, is assumed to be 0.32. The cavity loss is very small and is consistent with previous results [3]. The 3-DWELL lasers, having more quantum wells, have an improved internal efficiency compared to the 1-DWELL lasers.

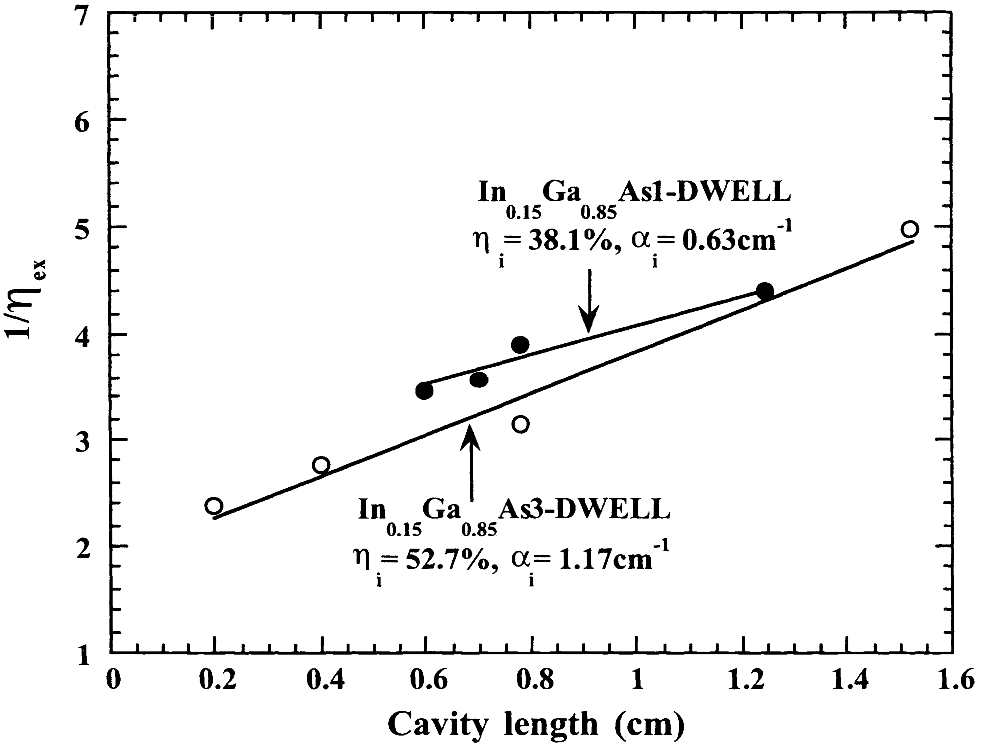

Fig.6. Inverse external quantum efficiency versus cavity length plot for the 1-DWELL and 3DWELL lasers

3.3. Modal gain

Using the results of threshold current density and internal loss, and the condition that gain is equal to loss at threshold, the ground state modal gains of the 1-DWELL and 3-DWELL lasers versus threshold current densities are plotted in Fig.7. By fitting the results using a formula $g=g_0 \ln (J/J_{tr})$, transparency current density of the DWELL structure may be estimated. The transparency current density is about 10.4 and 20.1 A cm^{-2} for 1-DWELL and 3-DWELL lasers, respectively. The maximum ground state gain for 3-DWELL lasers is improved to 12.5 cm^{-1} as compared to 3.6 cm^{-1} of 1-DWELL lasers. The improvement of ground state gain is due to both the increases of quantum dot number and the improved internal efficiency.

3.4 T_0 value

The threshold currents versus temperature of 1-DWELL, 3-DWELL, and 1-QW lasers with 7.8mm cavity length have been measured and are shown in Fig.8. The testing temperatures are varied from 10°C to 80°C. The results are fitted into a formula, $I_{th}=I_0 \text{Exp} (T/T_0)$. From fitting, T_0 values of 45 K, 84 K and 119 K are obtained for the 1-DWELL, 3-DWELL and 1-QW lasers. It should also be noticed that the fitting for the 1-DWELL lasers does not following a straight line in the linear-log plot. There is an up curvature, indicating that the T_0 value is decreasing with the increase of temperature. On the other hand, the fitting for the 3-DWELL lasers follows a straight line, and the T_0 value is constant over this temperature range. The 3-DWELL lasers show an improved temperature dependent threshold current performance.

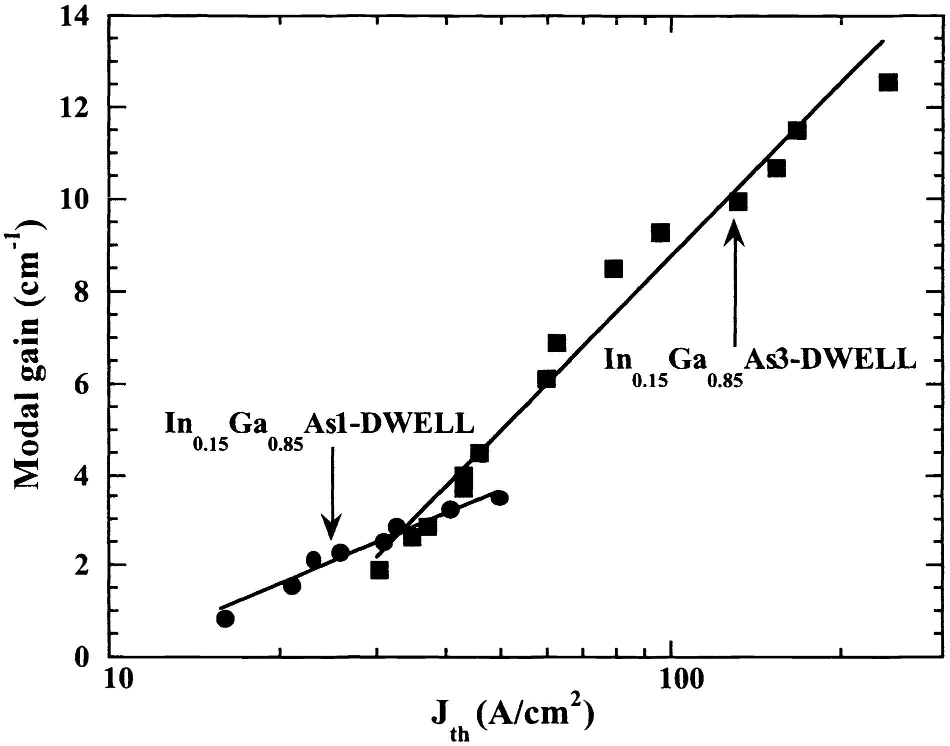

Fig.7. Linear-Log lot of ground state modal gain as a function of threshold current density for 1-DWELL and 3-DWELL laser

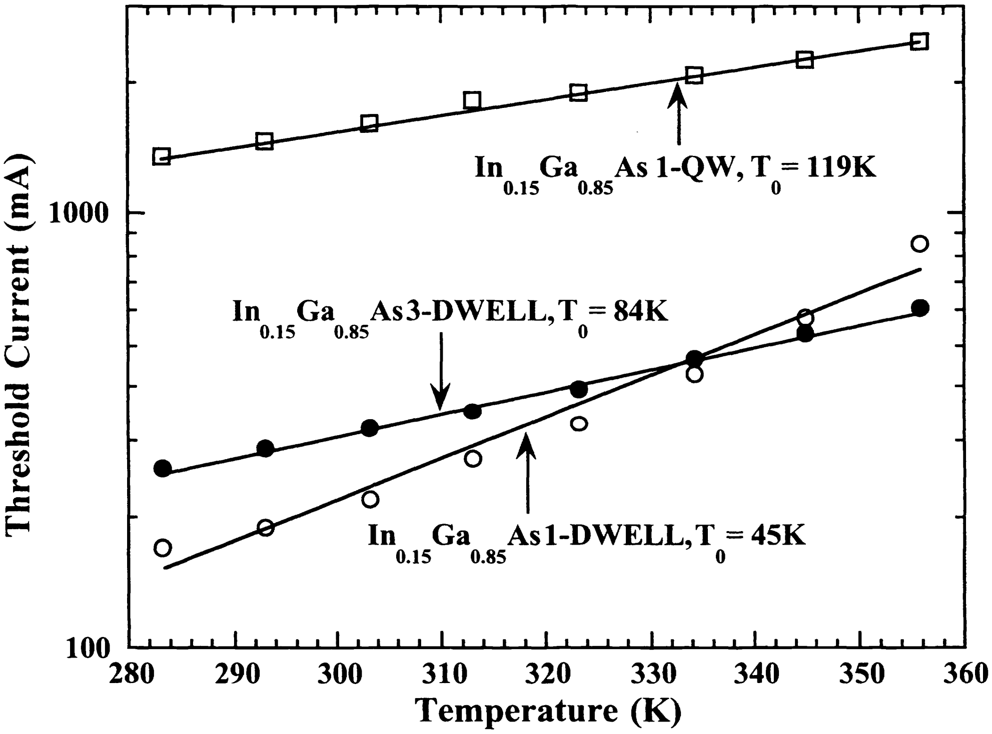

Fig.8. Plot of threshold current as a function of temperature for the 1-DWELL, 3-DWELL and 1-QW lasers.

It was original suspected that the T_0 value of the 1-DWELL structure was limited by the quantum well [2]. However, the 119K T_0 value of the 1-QW lasers is much higher than the 45K T_0 value of the 1-DWELL lasers. This means that the T_0 value of the 1-DWELL structure is limited by the quantum dot itself instead of by the quantum well. It is likely that T_0 value is limited by carrier heating from the quantum dots. The higher T_0 value of 84 K of the 3-DWELL laser supports this argument. When there are more quantum dots available, carrier heating effects are reduced.

4. CONCLUSIONS

Quantum dots lasers using the dots-in-a-well (DWELL) structure provide better carrier confinement, higher dot density and are more favorable for long wavelength lasers. As a result, threshold current densities as low as 16-26 A cm^{-2} have been achieved for a quantum dot laser at 1.25 μm with a single InAs dot layer in $In_{0.15}Ga_{0.85}As$ quantum well. This was the first time that the threshold current density of quantum dot laser was lower than any quantum well laser. These are also the lowest threshold current densities of any semiconductor laser. When using three DWELL stacks, the modal gain and internal quantum efficiency have also been improved. Carrier heating from the quantum dots is hypothesized to limit the T_0 value of the DWELL lasers.

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Effect of excited-state transitions on the threshold characteristics of a quantum dot laser

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ABSTRACT

Theoretical study of threshold characteristics of a quantum dot (QD) laser in the presence of excited-state transitions is given. The effect of microscopic parameters (degeneracy factor and overlap integral for a transition) on the gain is discussed. An analytical equation for the gain spectrum is derived in an explicit form. Transformation of the gain spectrum with the injection current is analyzed. The threshold current density is calculated as a function of the total losses. The conditions for a smooth or step-like change in the lasing wavelength with the losses are formulated. Threshold characteristics of a laser based on self-assembled pyramidal InAs QDs in GaAs matrix are simulated. A small overlap integral for transitions in such QDs (and hence large spontaneous radiative lifetime) is shown to be a main possible reason for a low value of the maximum single-layer modal gain of the respective structure which is deficient to attain lasing at moderately short (several hundreds of micrometers) cavity lengths.

Keywords: semiconductor heterojunctions, semiconductor lasers, quantum well and quantum dot lasers

1. INTRODUCTION

Quantum dot (QD) lasers become increasingly attractive as a novel type of injection lasers with enhanced characteristics.[1] There is a sufficient advance in fabricating such laser structures.[2–11]

Ideally, for the expected advantages of QD lasers over the conventional quantum well ones to be pronounced most strongly, there should be one electron and one hole energy level in a QD. For the QDs of highly symmetrical shape (e.g., cubic ones), the requirement is not so strong: it will suffice to have only one electron level in a QD. Due to the high-symmetry shape of such QDs, even though excited hole states exist, radiative transitions from the ground electron state to the excited hole ones are forbidden. In the actual laser structures, containing QDs of pyramidal shape, such transitions however are not forbidden. Besides, there may be also excited electron states in the QDs of actual sizes.[12] In what follows, transitions other than a transition from the ground electron state to that of a hole (referred to as a ground-state transition) are referred to as excited-state transitions. Excited-state transitions influence the threshold characteristics of a laser. In the article, a detailed theoretical treatment of this influence is given.

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2. GENERAL EQUATIONS

The modal gain spectrum is given as[13]

$$g(E) = \sum_{i,j} g_{ij}^{\max} \sqrt{\frac{\bar{\epsilon}_{ij}}{\epsilon(E)}} \, \frac{\Gamma(E)}{\bar{\Gamma}_{ij}} \, \frac{\bar{E}_{ij}}{E} \, \langle (f_{\mathrm{n},i} + f_{\mathrm{p},j} - 1) \, \delta\left(E - E_{ij}\right) \rangle \tag{1}$$

where $E_{ij} = E_{\mathrm{g}}^{\mathrm{QD}} + \varepsilon_{\mathrm{n},i} + \varepsilon_{\mathrm{p},j}$ is the energy of the transition from the i-th ($i = 1, 2, ...$) quantized energy level of an electron, $\varepsilon_{\mathrm{n},i}$, to the j-th ($j = 1, 2, ...$) quantized energy level of a hole, $\varepsilon_{\mathrm{p},j}$ (both measured from the corresponding band edges in a QD), and $E_{\mathrm{g}}^{\mathrm{QD}}$ is the QD bandgap. As noted in the Introduction, transitions with different indexes i and j would be forbidden in QDs of high-symmetry shape.

In (1), $f_{\mathrm{n},i}$ and $f_{\mathrm{p},j}$ are the i-th electron- and the j-th hole-level occupancies in QDs, the brackets $\langle ... \rangle$ mean averaging over the inhomogeneously broadened ensemble of QDs; $\bar{E}_{ij} = E_{\mathrm{g}}^{\mathrm{QD}} + \bar{\varepsilon}_{\mathrm{n},i} + \bar{\varepsilon}_{\mathrm{p},j}$, $\bar{\varepsilon}_{\mathrm{n},i}$ and $\bar{\varepsilon}_{\mathrm{p},j}$ are the i-th electron and j-th hole energy levels in a mean-sized QD.

The photon-energy dependent optical dielectric constant is denoted as $\epsilon = \epsilon(E)$, and $\bar{\epsilon}_{ij} = \epsilon(\bar{E}_{ij})$; $\Gamma = \Gamma(E)$ is the optical confinement factor in a QD layer (along the transverse direction in the waveguide) which also depends on the photon energy (through a such dependence of ϵ), and $\bar{\Gamma}_{ij} = \Gamma(\bar{E}_{ij})$.

The maximum modal gain corresponding to the transition from the i-th electron- to the j-th hole-level in a QD (referred to as the $i \to j$ transition) is

$$g_{ij}^{\max} = \frac{\xi}{4} \left(\frac{\bar{\lambda}_{ij}}{\sqrt{\bar{\epsilon}_{ij}}} \right)^2 d_{ij} \, \frac{1}{\tau_{ij}^{\mathrm{QD}}} \, \frac{\hbar}{(\Delta\varepsilon)_{\mathrm{inhom},ij}} \, \frac{\bar{\Gamma}_{ij}}{\bar{a}} \, N_{\mathrm{S}} \, Z_L \tag{2}$$

where ξ is a numerical constant appearing in QD size distribution function ($\xi = 1/\pi$ and $\xi = 1/\sqrt{2\pi}$ for the Lorentzian and Gaussian functions respectively), $\bar{\lambda}_{ij} = 2\pi\hbar c / \bar{E}_{ij}$, d_{ij} is the degeneracy factor of the $i \to j$ transition, $\bar{a}$ is the mean size of QDs, Z_{L} is the number of QD-layers, and N_{S} is the surface density of QDs in one layer.

The reciprocal of the spontaneous radiative lifetime for the $i \to j$ transition is[13,14,16]

$$\frac{1}{\tau_{ij}^{\mathrm{QD}}} = \frac{8}{3} \alpha \sqrt{\bar{\epsilon}_{ij}} \, \bar{\omega}_{ij} \left(\frac{P}{\hbar c} \right)^2 I_{ij} \tag{3}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant, $\bar{\omega}_{ij} = \bar{E}_{ij}/\hbar$, P is Kane's parameter, and I_{ij} is the overlap integral for the $i \to j$ optical transition.

Inhomogeneous broadening (caused by dispersion in QD sizes) of the line corresponding to the $i \to j$ transition is[13,14,16]

$$(\Delta\varepsilon)_{\mathrm{inhom},ij} = \left(q_{\mathrm{n},i} \, \bar{\varepsilon}_{\mathrm{n},i} + q_{\mathrm{p},j} \, \bar{\varepsilon}_{\mathrm{p},j} \right) \delta \tag{4}$$

where $q_{\mathrm{n},i} = -\left(\partial \ln \bar{\varepsilon}_{\mathrm{n},i} / \partial \ln \bar{a} \right)|_{a=\bar{a}}$, $q_{\mathrm{p},j} = -\left(\partial \ln \bar{\varepsilon}_{\mathrm{p},j} / \partial \ln \bar{a} \right)|_{a=\bar{a}}$, and δ is the root mean square (RMS) of relative QD size fluctuations.

In the case of equilibrium filling of QDs,[13]

$$f_{\mathrm{n},i} = \left[\exp\left(\frac{\varepsilon_{\mathrm{n},i} - \mu_{\mathrm{n}}}{T} \right) + 1 \right]^{-1} \qquad f_{\mathrm{p},j} = \left[\exp\left(\frac{\varepsilon_{\mathrm{p},j} - \mu_{\mathrm{p}}}{T} \right) + 1 \right]^{-1} \tag{5}$$

where μ_{n} and μ_{p} are the electron and hole quasi-Fermi levels (measured from the corresponding band edges in a QD), and the temperature T being measured in terms of energy.

Averaging in eq. (1) gives an exact analytical equation for the gain spectrum (see eq. (18) in Ref.[13]). To use this equation however requires a knowledge of the dependence of the quantized energy levels on the QD size. In the case of relatively small QD-size fluctuations around their mean values, the energy levels in a mean-sized QD, $\bar{\varepsilon}_{\mathrm{n},i}$